Integer Solutions to Surd Equation with Five Unknowns

$$\sqrt[3]{x^2+y^2}+\sqrt[2]{X^2+2Y^2}=35z^3$$

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Abstract

The transcendental equation involving surd with five variables given by

$\sqrt[3]{x^2+y^2}+\sqrt[2]{X^2+2Y^2}=35z^3$ is studied for obtaining its integer solutions.

Substitution technique and factorization method are utilized to obtain the required integer solutions.

Keywords: Surd equation, Transcendental equation, integer solutions, Quinary surd Equation, Transformations, Factorization method

Introduction

The subject of Diophantine equation, one of the interesting areas of Number Theory, plays a significant role in higher arithmetic and has a marvellous effect on credulous people and always occupies a remarkable position due to unquestioned historical importance. The Diophantine equations may be either polynomial equation with at least two unknowns for which integer solution, are required or transcendental equation involving trigonometric, logarithmic, exponential and surd function such that one may be interested in getting integer solution. Most of the Diophantine problems solved by the researchers are polynomial equations.[1-13]. Note that, the transcendental equation can be solved by transforming it into an equivalent polynomial equation. Adhoc methods exists for some classes of transcendental equations in one variable to transform them into polynomial equations which then might be solved. Some transcendental equation in more than one unknown can be solved by separation of the unknowns reducing them to polynomial equations. In this context, one may refer [14-18].

In this paper, we are interested in obtaining integer solutions to transcendental equation involving surds. In particular, we obtain different sets of integer solutions to the transcendental equation with five unknowns given by $\sqrt[3]{x^2+y^2}+\sqrt[2]{X^2+2Y^2}=35z^3$. Substitution technique and factorization method are utilized to obtain the required integer solutions.

Method of analysis

The quinary surd equation to be solved is

$$\sqrt[3]{x^2+y^2}+\sqrt[2]{X^2+2Y^2}=35z^3 \tag{1}$$

The insertion of the transformations

$$x=u(u^2+v^2),\, y=v(u^2+v^2),\, X=(u^2-2v^2),\, Y=2uv \tag{2}$$

in (1) leads to non-homogeneous ternary cubic equation

$$2u^2+3v^2=35z^3 \tag{3}$$

Again , taking

$$u=p+3q,\, v=p-2q \tag{4}$$

in (3) , one obtains

$$p^2+6q^2=7z^3 \tag{5}$$

Solving (5) through different ways and using (2) ,patterns of integer solutions

to (1) are obtained.

Pattern 1

Let

$$z=A^2+6B^2 \tag{6}$$

Write the integer 7 in (5) as

$$7=(1+i\sqrt{6})(1-i\sqrt{6}) \tag{7}$$

Substituting (6) & (7) in (5) and employing factorization , consider

$$p+i\sqrt{6}\,q=(1+i\sqrt{6})\,(A+i\sqrt{6}B)^3=(1+i\sqrt{6})\,[f(A,B)+i\sqrt{6}\,g(A,B)] \tag{8}$$

where

$$\begin{aligned} f(A,B) &= A^3 - 18AB^2 \\ g(A,B) &= 3A^2B - 6B^3 \end{aligned} \tag{9}$$

On comparing the coefficients of corresponding terms in (8) ,we get

$$\begin{aligned} p &= f(A,B) - 6g(A,B) \\ q &= f(A,B) + g(A,B) \end{aligned}$$

In view of (4) , we have

$$\begin{aligned} u &= 4f(A,B) - 3g(A,B), \\ v &= -f(A,B) - 8g(A,B). \end{aligned}$$

From (2) ,we obtain

$$\begin{aligned} x &= [4f(A,B) - 3g(A,B)]\{17(f(A,B))^2 + 73(g(A,B))^2 - 8f(A,B)*g(A,B)\} \\ y &= -[f(A,B) + 8g(A,B)]\{17(f(A,B))^2 + 73(g(A,B))^2 - 8f(A,B)*g(A,B)\} \\ X &= 14(f(A,B))^2 - 119(g(A,B))^2 - 56f(A,B)*g(A,B) \\ Y &= -2[4(f(A,B))^2 - 24(g(A,B))^2 + 29f(A,B)*g(A,B)] \end{aligned} \tag{10}$$

Thus,(6)&(10) satisfy (1).

Some numerical solutions to (1) are given below:

Example 1

$$\begin{aligned} &A = B = 1 \\ &f(1,1) = -17,\ g(1,1) = -3 \\ &x = -59*5162,\ y = 41*5162,\ X = 119,\ Y = -4838,\ z = 7 \end{aligned}$$

Example 2

$$\begin{aligned} &A = 2, B = 1 \\ &f(2,1) = -28,\ g(2,1) = 6 \\ &x = -130*17300,\ y = -20*17300,\ X = 16100,\ Y = 5200,\ z = 10 \end{aligned}$$

Note 1

Apart from (7) , one may also take

$$7=\frac{(13+i\sqrt{6})\,(13-i\sqrt{6})}{25}$$
$$7=\frac{(-13+i\sqrt{6})\,(-13-i\sqrt{6})}{25}$$
$$7=(-1+i\sqrt{6})\,(-1-i\sqrt{6})$$

Following the above process , one obtains three more sets of integer solutions to (1).

Pattern 2

Write (5) as

$$p^2+6q^2=7z^3*1 \tag{11}$$

Consider the integer 1 in (11) as

$$1=\frac{(1+i2\sqrt{6})\,(1-i2\sqrt{6})}{25} \tag{12}$$

Take

$$z=25(A^2+6B^2) \tag{13}$$

Inserting (7) ,(12) & (13) in (11) and applying factorization , consider

$$p+i\sqrt{6}\,q=(1+i\sqrt{6})\;*\,5^3[f(A,B)+i\sqrt{6}\,g(A,B)]*\frac{[1+i\;2\sqrt{6}]}{5}$$
$$=(-11+i\;3\sqrt{6})\;*\,5^2[f(A,B)+i\sqrt{6}\,g(A,B)]$$

from which we have

$$p=-5^2\{11\,f(A,B)+18\,g(A,B)\}$$
$$q=5^2\{3\,f(A,B)-11\,g(A,B)\}$$

In view of (4) ,we have

$$u=-5^2\{2\,f(A,B)+51\,g(A,B)\}$$
$$v=5^2\{-17\,f(A,B)+4\,g(A,B)\}$$

In view of (2) ,the integer solutions to (1) are obtained as

$$x=-5^{6}[2f(A,B)+51g(A,B)]\{293(f(A,B))^{2}+2617(g(A,B))^{2}+68f(A,B)*g(A,B)\}$$
$$y=5^{6}[-17f(A,B)+4g(A,B)]\{293(f(A,B))^{2}+2617(g(A,B))^{2}+68f(A,B)*g(A,B)\}$$
$$X=5^{4}[-574f(A,B))^{2}+2569(g(A,B))^{2}+476f(A,B)*g(A,B)]$$
$$Y=-2*5^{4}[-34(f(A,B))^{2}+204(g(A,B))^{2}-859f(A,B)*g(A,B)]$$

jointly with (13).

Example 3

$$A=B=1$$
$$f(1,1)=-17\,,\,g(1,1)=-3$$
$$x=5^{6}*187*111698,\ y=5^{6}*277*111698,\ X=-5^{4}*118489,\ Y=2*5^{4}*187*277\,,\ z=175$$

Note 2

In addition to (12) , the integer 1 in (11) is expressed as below:

$$1=\frac{(2n^{2}-3+i2n\sqrt{6})(2n^{2}-3-i2n\sqrt{6})}{(2n^{2}+3)^{2}}$$
$$1=\frac{(3n^{2}-2+i2n\sqrt{6})(3n^{2}-2-i2n\sqrt{6})}{(3n^{2}+2)^{2}}$$
$$1=\frac{(6r^{2}-s^{2}+i\,(2rs)\sqrt{6})\,(6r^{2}-s^{2}-i\,(2rs)\sqrt{6})}{(6r^{2}+s^{2})^{2}}$$
$$1=\frac{(r^{2}-6s^{2}+i\,(2rs)\sqrt{6})\,(r^{2}-6s^{2}-i\,(2rs)\sqrt{6})}{(r^{2}+6s^{2})^{2}}$$

Following the above procedure , one obtains four more sets of integer solutions to (1).

Pattern 3

It is seen that (5) is satisfied by

$$p=7^{2}P(P^{2}+6Q^{2})\,,q=7^{2}Q(P^{2}+6Q^{2})$$

and

$$z=7(P^{2}+6Q^{2}) \tag{14}$$

From (4) , we have

$$u=7^{2}(P^{2}+6Q^{2})\,(P+3Q)$$
$$v=7^{2}(P^{2}+6Q^{2})\,(P-2Q)$$

In view of (2) , the integer solutions to (1) are given by

$$x = 7^6 (P+3Q)(P^2+6Q^2)^3 [2P^2+13Q^2+2PQ]$$
$$y = 7^6 (P-2Q)(P^2+6Q^2)^3 [2P^2+13Q^2+2PQ]$$
$$X = 7^4 (P^2+6Q^2)^2 [-P^2+Q^2+14PQ]$$
$$Y = 2*7^4 (P^2+6Q^2)^2 [P^2-6Q^2+PQ]$$

jointly with (14).

Example 4

$P = Q = 1$

$x = 68*7^9, y = -17*7^9, X = 2*7^7, Y = -8*7^6, z = 7^2$

Pattern 4

The choice

$$q = z \tag{15}$$

in (5) gives

$$p^2 = z^2 (7z-6) \tag{16}$$

After some algebra ,it is seen that (16) is satisfied by

$$z = 7k^2 - 12k + 6 \tag{17}$$

and

$$p = (7k-6)(7k^2-12k+6)$$

In view of (4) , we have

$$u = (7k-3)(7k^2-12k+6)$$
$$v = (7k-8)(7k^2-12k+6)$$

In view of (2) , the integer solutions to (1) are given by

$$x = (7k-3)(7k^2-12k+6)^3 [(7k-3)^2 + (7k-8)^2]$$
$$y = (7k-8)(7k^2-12k+6)^3 [(7k-3)^2 + (7k-8)^2]$$
$$X = (7k^2-12k+6)^2 [(7k-3)^2 - 2(7k-8)^2]$$
$$Y = 2(7k^2-12k+6)^2 (7k-3)(7k-8)$$

jointly with (17).

Pattern 5

Let

$$\alpha^2 = 7z - 6 \tag{18}$$

which is satisfied by

$$z_0 = 7k^2 - 12k + 6, \alpha_0 = 7k - 6 \tag{19}$$

Assume the second solution to (18) as

$$\alpha_1 = h - \alpha_0 \,, z_1 = h + z_0 \tag{20}$$

where h is an unknown to be determined. Substituting (20) in (18) and simplifying , we have

$$h = 2\alpha_0 + 7$$

and in view of (20) ,it is seen that

$$\alpha_1 = \alpha_0 + 7, z_1 = z_0 + 2\alpha_0 + 7$$

The repetition of the above process leads to the general solution to (18) as

$$\begin{aligned} \alpha_n &= \alpha_0 + 7\,n \\ z_n &= z_0 + 2n\alpha_0 + 7\,n^2 = q_n \end{aligned} \tag{21}$$

From (16) ,we have

$$p_n = z_n * \alpha_n$$

In view of (4) , we get

$$u_n = z_n(\alpha_n + 3)\,, v_n = z_n(\alpha_n - 2)$$

In view of (2) , the integer solutions to (1) are given by

$$\begin{aligned} x_n &= (z_n)^3(\alpha_n + 3)[\,(\alpha_n + 3)^2 + (\alpha_n - 2)^2] \\ y_n &= (z_n)^3(\alpha_n - 2)[\,(\alpha_n + 3)^2 + (\alpha_n - 2)^2] \\ X_n &= (z_n)^2[\,(\alpha_n + 3)^2 - 2\,(\alpha_n - 2)^2] \\ Y_n &= 2\,(z_n)^2\,(\alpha_n + 3)\,(\alpha_n - 2) \end{aligned}$$

jointly with z_n in (21) .

Remark

It is worth to mention that , apart from (4) , one may choose $u = p - 3q\,, v = p + 2q$ giving different sets of integer solutions to (1).

Conclusion

In this paper, the transcendental equation involving surds with five unknowns given by $\sqrt[3]{x^2 + y^2} + \sqrt[2]{X^2 + 2Y^2} = 35\,z^3$ has been studied to obtain integer solutions . in an elegant way through suitable transformations and utilizing the technique of factorization. As surd equations are plenty , one may attempt for getting integer solutions to other choices of surd equations with more variables.

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